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The Extremal Solutions of the Equation $Ly + p(x)y = 0$

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1. INTRODUCTION

In this paper we consider the differential equation

$$Ly + p(x)\, y = 0, \tag{1}$$

where L is a disconjugate linear operator of order n and $p(x)$ is a continuous function in $[0, \infty)$. By the well known theorem of Polya [8], we may assume that the operator L is given by the factorization

$$L = \rho_{n+1}\, D\rho_n \cdots \rho_2\, D\rho_1\,,$$

where $\rho_i \in C^{n-i+1}$ and $\rho_i > 0$ in $[0, \infty)$, $i = 1,..., n+1$. For short we denote $L_0 y = \rho_1 y$ and $L_i y = \rho_{i+1}\, D(L_{i-1} y)$, $i = 1,..., n-1$.

Let us assume that there is a nontrivial solution of Eq. (1) which vanishes at a and has at least $n + k - 1$ zeros, $k = 1, 2,...$, in $[a, x]$, $x > a$. The infimum of points x which has this property, exists. It is called the k-th conjugate point of a and is denoted by $\eta_k(a)$. Using compactness argument ([1]), one may easily show that if $\eta_k(a)$ exists then $\eta_k(a) > a$ and there is a solution which vanishes at a and at $\eta_k(a)$ and has at least $n + k - 1$ zeros in $[a, \eta_k(a)]$. Such a solution is called an extremal solution for the interval $[a, \eta_k(a)]$.

The distribution of zeros of extremal solutions was investigated by Leighton and Nehari [4] for the equation

$$(ry'')'' + p(x)\, y = 0.$$

Hunt [2] considered the distribution of the zeros of solutions of the self-adjoint differential equation

$$(ry^{(n)})^{(n)} + p(x)\, y = 0.$$

Johnson [3] studied the same problem for Eq. (1) when L is an even order operator and $p(x) \leqslant 0$. In this paper, the order n of the operator L is arbitrary and we assume that $p(x)$ has a constant sign. The main result is the following:

THEOREM 1. *Let $y(x)$ be an extremal solution of Eq.* (1) *for the interval* $[a, \eta_k(a)]$. *Then $y(x)$ has exactly $n + k - 1$ zeros in $[a, \eta_k(a)]$. The only zeros of $y(x)$ in $(a, \eta_k(a))$ are exactly $k - 1$ zeros of odd multiplicities. The zero at $\eta_k(a)$ is of odd or even multiplicity according to whether $p(x) \geqslant 0$ or $p(x) \leqslant 0$. If $p(x) \leqslant 0$ then $y(x)$, $L_1 y, \ldots, L_{n-1} y$ have no zeros in $(\eta_k(a), \infty)$ and if $(-1)^n p(x) \leqslant 0$, then similar conclusion holds for $[0, a)$.*

This theorem will be used to establish the existence of solutions with given number of simple zeros and to prove some properties of $\eta_k(a)$ as a function of a.

2.

DEFINITION. Let $y(x)$ be a solution of Eq. (1) which has at the points $x_1, \ldots, x_r$, $a = x_1 < \cdots < x_r = b$, zeros of multiplicities $m(x_1), \ldots, m(x_r)$, respectively. For the solution $y(x)$ and the interval $[a, b]$ we define (cf. [3]),

$$I = \{i \mid m(x_i) \text{ is even or } x_i = a \text{ or } x_i = b\},$$

$$J = \{j \mid a < x_j < b \text{ and } m(x_j) \text{ is odd}\},$$

$$M(y) = \sum_{i \in I} m(x_i) + \sum_{j \in J} [m(x_j) - 1]$$

We shall denote by $m(x_t, y)$ the multiplicity of the zero of the solution $y(x)$ at x_t.

For the first two lemmas there is no need to assume that $p(x)$ is of constant sign, so we assume that $p(x)$ changes its sign l times in (a, b).

LEMMA 1. *Every solution of Eq.* (1) *satisfies $M(y) \leqslant n + l$. Moreover, if $M(y) = n + l$, then $L_t y$ $(t = 1, \ldots, n - 1)$ has zeros only of the following types*:

(a) *A zero at a point where $y(x)$ has a zero of multiplicity bigger than t.*

(b) *Exactly one simple zero between consecutive zeros of $L_{t-1} y$.*

Proof. (Cf. [2]). We assume that $y(x)$ has at the points $a = x_1 < \cdots < x_r = b$ zeros of multiplicities $m(x_1), \ldots, m(x_r)$. We denote the number of zeros of multiplicity bigger or equal to i by r_i. Evidently, $r_1 = r$ and $r_n = 0$. Now, $\sum_{i=1}^{n-1} r_i$ is the number of the zeros of $y(x)$ in $[a, b]$ counting multiplicities, since in the summation the zero at x_i is counted exactly $m(x_i)$ times.

$L_0 y = \rho_1 y$ vanishes at r_1 different points of $[a, b]$. $L_1 y$ vanishes by Rolle's theorem at least at $r_1 - 1$ points between the r_1 points where $L_0 y$ vanishes, and at r_2 points where $y(x)$ has a zero of multiplicity bigger than 1. Therefore $L_1 y$ vanishes at least at $r_1 + r_2 - 1$ different points of $[a, b]$. It follows similarly that $L_{n-2} y$ vanishes at least at $r_1 + \cdots + r_{n-1} - (n-2)$ different points of $[a, b]$. By Rolle's theorem, $L_{n-1} y$ changes its sign at least $r_1 + \cdots + r_{n-1} - (n-1)$ times in (a, b), and Ly changes its sign at least $\sum_{i=1}^{n-1} r_i - n$ times.

Therefore $p(x) y = -Ly$ changes its sign at least at $\sum_{i=1}^{r} m(x_i) - n$ different points of (a, b). Now, $p(x)$ changes its sign l times in (a, b) and $y(x)$ changes its sign at $|J| = \sum_{j \in J} 1$ points. Hence,

$$\sum_{I \cup J} m(x_i) - n \leqslant \sum_{J} 1 + l,$$

and the inequality $M(y) \leqslant n + l$ follows.

The equality $M(y) = n + l$ occurs if and only if $L_t y$ $(t = 1, \ldots, n-1)$ vanishes exactly at $r_1 + \cdots + r_{t+1} - t$ different points. Among these points there are exactly r_{t+1} points where $y(x)$ has a zero of multiplicity bigger than t and exactly $r_1 + \cdots + r_t - t$ zeros which are located by Rolle's theorem between the $r_1 + \cdots + r_t - (t-1)$ points where L_{t-1} vanishes. Now the zeros which exist according to Rolle's theorem are simple. For, $L_{t+1} y$ vanishes at r_{t+2} points where $y(x)$ has zeros of multiplicities bigger than $t + 1$, and at $r_1 + \cdots + r_{t+1} - (t+1)$ points between the zeros of $L_t y$. If one of the $r_1 + \cdots + r_t - t$ zeros of $L_t y$ which exists according to Rolle's theorem were a multiple zero, then L_{t+1} would have an additional zero at this point. This is impossible since $L_{t+1} y$ vanishes exactly at $r_1 + \cdots + r_{t+2} - (t+1)$ points.

Thus multiple zeros of $L_t y$ $(t = 1, \ldots, n-1)$ are located only at points where the preceeding derivatives have multiple zeros. In particular $L_{n-2} y$ has exactly $\sum_{i=1}^{r} m(x_i) - (n-2)$ simple zeros in $[a, b]$ since the zeros of $y(x)$ are of multiplicity less than n. $L_{n-1} y$ has therefore exactly $\sum_{i=1}^{r} m(x_i) - (n-1)$ simple zeros, all of them in (a, b). Ly changes its sign by Rolle's theorem exactly $\sum_{i=1}^{r} m(x_i) - n$ times. Of course, Ly may have even order zeros in (a, b).

Lemma 2. *Let $y(x)$ be a solution of Eq.* (1) *which satisfies $M(y) = n + l$ in $[a, b]$. If $p(x) < 0$ ($p(x) > 0$) in a left neighborhood of b, then $m(b)$ and $n + l - m(a)$ are even* (*odd*).

Proof. By Lemma 1, $L_t y$ does not vanish right of the last zero of $L_{t-1} y$ in $[a, b]$. Especially,

$$(L_0 y)(b) = \cdots = (L_{m(b)-1} y)(b) = 0,$$

and

$$(L_{m(b)}y)(b) \neq 0,\dots, (L_{n-1}y)(b) \neq 0.$$

We denote the last zero of $L_t y(m(b) \leqslant t \leqslant n-1)$ in $[a, b]$ by β_t.

By the previous observation

$$\beta_{n-1} \leqslant \beta_{n-2} \leqslant \cdots \leqslant \beta_{m(b)} < b.$$

The last zero of Ly in $[a, b]$ is of course b. But if we denote the last of the $\sum_{i=1}^{r} m(x_i) - n$ changes of sign of Ly by β_n, then $\beta_n < \beta_{n-1}$. Hence $p(x)\,y(x)$ has a constant sign in $[\beta_n, b]$ and especially in $[\beta_{n-1}, b]$.

We consider the case when $p(x) < 0$ in some left neighborhood of b. Without loss of generality we may assume that $y(x) > 0$ left to b, i.e., $(-1)^{m(b)}(L_{m(b)}y)(b) > 0$.

Integrating $\rho_{n+1}(x)\, D(L_{n-1}y) = -p(x)\,y(x)$ on (β_{n-1}, b), we obtain

$$(L_{n-1}y)\,(b) = L_{n-1}y\,\Big|_{\beta_{n-1}}^{b} = \int_{\beta_{n-1}}^{b} -\frac{p(x)\,y(x)}{\rho_{n+1}(x)}\,dx > 0.$$

Therefore $L_{n-1}y > 0$ in $(\beta_{n-1}, b]$ and especially in $(\beta_{n-2}, b]$. By integrating $\rho_n(x)\,D(L_{n-2}y) = L_{n-1}y$ on (β_{n-2}, b), we obtain that $L_{n-2}y > 0$ on $(\beta_{n-2}, b]$. In a similar way we obtain that $(L_{m(b)}y)(b) > 0$. In view of $(-1)^{m(b)}(L_{m(b)}y)(b) > 0$, we deduce that $m(b)$ is even. As $M(y) = n + l$ and as $m(a) + m(b) \equiv M(y) \pmod 2$, it follows that $m(b)$ and $n + l - m(a)$ are of the same parity.

When $p(x) > 0$ in some left neighborhood of b, the proof is similar. The proof of the lemma is valid even if $p(x)$ is not continuous at b but $p(x)\,y(x)$ is integrable near b.

COROLLARY 1. *Any oscillatory solution of Eq.* (1) *has a finite number of multiple zeros and infinitely many simple zeros.*

This follows readily from the boundedness of $M(y)$.

In the remainder of the paper we assume that $p(x)$ is of a constant sign. For convenience we restate Lemma 1 and Lemma 2 for that case.

LEMMA 3. *Let $p(x)$ be of a constant sign. For every solution $y(x)$ of Eq.* (1), *$M(y) \leqslant n$. If $M(y) = n$ then $m(b)$ and $n - m(a)$ are odd* (*even*) *when $p(x) \geqslant 0$ ($p(x) \leqslant 0$).*

COROLLARY 2. *Let $p(x) \leqslant 0$ and let $y(x)$ be a solution of Eq.* (1) *such that $M(y) = n$ in $[a, b]$. Then none of the functions $L_0y, L_1y,\dots, L_{n-1}y$ vanishes in (b, ∞) and all of them have the same sign. For the special equation*

$y^{(n)} + p(x)\,y = 0, y, y',\ldots, y^{(n-1)}$ are all monotone and $|\,y(x)| \geqslant Ax^{n-1}$ when $x \to \infty$.

Proof. There is some neighborhood of b such that $L_t y$ $(0 \leqslant t \leqslant n-1)$ do not vanish in it. Let $c > b$ be the first point to the right of b such that one of the derivatives, say $L_s y$, vanishes at c. By the proof of Lemma 1, $L_s y$ vanishes at $r_1 + \cdots + r_{s+1} - s$ different points of $[a, b]$ and also at c. We consider now Eq. (1) in $[a, c]$. $L_s y$ vanishes at $r_1 + \cdots + r_{s+1} - s + 1$ different points of $[a, c]$ and therefore Ly changes its sign at least $\sum_{i=1}^{r} m(x_i) - n + 1$ times in (a, c), i.e., more times than in (a, b). But this is impossible since $m(b)$ is even and $p(x)\,y(x)$ does not change its sign in $(b - \epsilon, c)$.

If $y(x) > 0$ in (b, ∞), then by the proof of Lemma 2 we see that $(L_t y)(b) \geqslant 0$ $(0 \leqslant t \leqslant n-1)$ and therefore all the functions $L_t y$ are strictly increasing. For the equation $y^{(n)} + p(x)\,y = 0$ we obtain by integration

$$y(x) \geqslant \sum_{t=m(b)}^{n-1} \frac{1}{t!}\, y^{(t)}(b)\,(x - b)^t.$$

If $(-1)^n\, p(x) \leqslant 0$, similar properties can be proved for $[0, a)$.

Remark. If $p(x)$ changes its sign at l points, all of them in (a, b), and $p(x) < 0$ near b, then the same property holds for every solution of Eq. (1) which satisfies $M(y) = n + l$. The proof is identical to that of Corollary 2.

COROLLARY 3. *The sum of the multiplicities of the zeros of a solution of Eq.* (1) *at s points does not exceed $n + s - 2$.*

This follows by Lemma 3, since $\sum_{i=1}^{s} m(x_i) \leqslant M(y) + (s - 2)$. For $s = 2$, this was proved by Nehari [7], by applying generalized Wronskians. As a matter of fact, Theorems 3.2–3.4 of [7] may be proved by applying Lemma 3. Also Theorem 4 and 8 of Levin [5] are particular cases of Lemma 1 and Lemma 2.

Now we turn to the extremal solutions of Eq. (1). Like Johnson [3, Lemma 3] we prove the following.

LEMMA 4. *If $y(x)$ is an extremal solution for $[a, \eta_k(a)]$, then $M(y) = n$.*

Proof. We prove that if $y(x)$ is a solution of Eq. (1) in $[a, b]$ such that $M(y) < n$, then there exists another solution of Eq. (1) with the same number of zeros in $[a, b)$, hence $y(x)$ is not an extremal solution.

First, we assume that $p(x) \geqslant 0$. Let $y(x)$ have at the points $a = x_1 < \cdots < x_r = b$ zeros of multiplicities $m(x_1),\ldots, m(x_r)$ such that $M(y) < n$.

We consider the following $M(y)$ boundary value conditions

$$
\begin{aligned}
u^{(t)}(x_i) &= 0, \qquad 0 \leqslant t \leqslant m(x_i) - 1, \qquad i \in I, \quad x_i \neq b,\\
u^{(t)}(x_j) &= 0, \qquad 0 \leqslant t \leqslant m(x_j) - 2, \qquad j \in J,\\
u^{(t)}(b) &= 0, \qquad 0 \leqslant t \leqslant m(b) - 2,\\
u^{(m(b)-1)}(b) &= 1,
\end{aligned}
$$

and add $n - M(y) \geqslant 1$ conditions. If $n - m(a)$ is even, we add $n - M(y)$ conditions at b:

$$u^{(t)}(b) = 0, \qquad m(b) \leqslant t \leqslant m(b) + n - M(y) - 1.$$

If $n - m(a)$ is odd, we add one condition at a and $n - M(y) - 1$ conditions at b:

$$
\begin{aligned}
u^{(m(a))}(a) &= 0,\\
u^{(t)}(b) &= 0, \qquad m(b) \leqslant t \leqslant m(b) + n - M(y) - 2.
\end{aligned}
$$

This nonhomogeneous system of n boundary value conditions will be denoted by (B). Now, the associated homogeneous system (H) has only the trivial solution. Indeed, if (H) had a nontrivial solution $v(x)$, then $M(v) = n$ and $n - m(a, v)$ would be even, thus contradicting Lemma 3. Therefore the nonhomogeneous system (B) has a unique solution which is denoted by $\bar{y}(x)$. $\bar{y}(x)$ has at b a zero exactly of multiplicity $m(b, y) - 1$ and $\bar{y}(x)$ and $y(x)$ are thus linearly independent.

For every α, the solution $y_1(x) = y(x) + \alpha\bar{y}(x)$ has at x_i ($i \in I$, $x_i \neq b$ a zero at least of multiplicity $m(x_i)$, at x_j ($j \in J$) a zero at least of multiplicity $m(x_j) - 1$, and at b a zero exactly of multiplicity $m(b) - 1$. By Taylor's theorem it follows that for $|\alpha|$ sufficiently small, $y_1(x)$ has additional zeros in given neighborhoods of x_j ($j \in J$) (possibly at x_j) and b. These zeros are simple for small α. Else, as $\alpha \to 0$, we would find that $y(x) = \lim_{\alpha\to 0} y_1(x)$ had at x_j a zero of multiplicity greater than $m(x_j)$. Moreover, if the sign of α is properly chosen, the simple zero near b will be to the left of b, in $[a, b)$. $y_1(x)$ has in $[a, b]$ the same number of zeros as $y(x)$ and $M(y_1) < M(y)$.

By repeating a similar process $m(b)$ times, we shift the zeros at b one by one to the left and obtain a solution which has in $[a, b)$ the same number of zeros as $y(x)$ has in $[a, b]$.

When $p(x) \leqslant 0$ the proof is similar.

LEMMA 5. *An extremal solution for $[a, \eta_k(a)]$ has exactly $n + k - 1$ zeros in this interval.*

Proof. Let us assume on the contrary that an extremal solution $y(x)$ has at least $n + k$ zeros in $[a, \eta_k(a)]$. The zero of $y(x)$ at $\eta_k(a)$ is not simple, otherwise $y(x)$ would have $n + k - 1$ zeros in $[a, \eta_k(a))$. We consider now the following $M(y) - 2 = n - 2$ boundary value conditions:

$$\begin{aligned} u^{(t)}(x_i) &= 0, \qquad 0 \leqslant t \leqslant m(x_i) - 1, \qquad i \in I, \quad x_i \neq \eta_k(a), \\ u^{(t)}(\eta_k(a)) &= 0, \qquad 0 \leqslant t \leqslant m(\eta_k(a)) - 3, \\ u^{(t)}(x_j) &= 0, \qquad 0 \leqslant t \leqslant m(x_j) - 2, \qquad j \in J. \end{aligned}$$

This problem has a nontrivial solution $\bar{y}(x)$, linearly independent of $y(x)$.

Assume that $\bar{y}(x)$ has at $\eta_k(a)$ a zero exactly of multiplicity $m(\eta_k(a)) - 2$. Then for $|\alpha|$ sufficiently small and α of suitable sign, the solution $y_1(x) = y(x) + \alpha\bar{y}(x)$ has at $\eta_k(a)$ a zero of multiplicity $m(\eta_k(a)) - 2$ and two simple zeros to the right and to the left of $\eta_k(a)$. Moreover, $y_1(x)$ has simple zeros near each x_j $(j \in J)$. By simple count we find that $y(x)$ has at least $n + k - 1$ zeros in $[a, \eta_k(a)]$ and $M(y_1) = n - 2$, contradicting Lemma 4.

If $\bar{y}(x)$ has at $\eta_k(a)$ a zero of multiplicity $m(\eta_k(a)) - 1$, then similarly $y_1(x) = y(x) + \alpha\bar{y}(x)$ has at least $n + k - 1$ zeros in $[a, \eta_k(a)]$ and $M(y_1) = n - 1$.

If $\bar{y}(x)$ has at $\eta_k(a)$ a zero of multiplicity $m(\eta_k(a))$ or more, then there is a linear combination of $\bar{y}(x)$ and $y(x)$ such that $M(c_1 y + c_2\bar{y}) > n$, yielding again a contradiction.

As a result of Lemma 5, we conclude that $\eta_k(a) < \eta_{k+1}(a)$.

LEMMA 6. *Every extremal solution for $[a, \eta_k(a)]$ has exactly $k - 1$ zeros of odd multiplicity and no zero of even multiplicity in $(a, \eta_k(a))$.*

Proof. Every extremal solution for $[a, \eta_k(a)]$ satisfies

$$\sum_{I \cup J} m(x_t) = n + k - 1, \qquad \sum_I m(x_i) + \sum_J [m(x_j) - 1] = n.$$

Therefore $|J| = k - 1$ and every extremal solution has exactly $k - 1$ zeros of odd multiplicity in $(a, \eta_k(a))$. Assume now that an extremal solution $y(x)$ has in $(a, \eta_k(a))$ a zero x_s of even multiplicity ($s \in I$, $x_s \neq a$, $\eta_k(a)$, $m(x_s) \geqslant 2$). In the system

$$\begin{aligned} u^{(t)}(x_i) &= 0, \qquad 0 \leqslant t \leqslant m(x_i) - 1, \qquad i \in I, \quad i \neq s, \\ u^{(t)}(x_s) &= 0, \qquad 0 \leqslant t \leqslant m(x_s) - 3, \\ u^{(t)}(x_j) &= 0, \qquad 0 \leqslant t \leqslant m(x_j) - 2, \qquad j \in J, \end{aligned}$$

there are $\sum_{i\neq s} m(x_i) + [m(x_s) - 2] + \sum_J [m(x_j) - 1] = M(y) - 2 = n - 2$ boundary value conditions. This system has a nontrivial solution $\bar{y}(x)$ linearly independent of $y(x)$. We now obtain the desired contradiction as in Lemma 5, by considering the multiplicity of the zero of $\bar{y}(x)$ at x_s.

Lemmas 5, 6 and 3 and Corollary 2 yield the assertions of Theorem 1.

Remark. It is well known [7, 6] that every linear differential equation of order n has an extremal solution for $[a, \eta_1(a)]$ which is positive in $(a, \eta_1(a))$. For Eq. (1), *every* extremal solution for $[a, \eta_1(a)]$ has this property.

3.

THEOREM 2. *If a solution of Eq.* (1) *has m ($m \geqslant n$) zeros (counting multiplicities) in an open or a half open interval, then there exists a solution of Eq.* (1) *with at least m simple zeros in the same interval.*

Proof. If a solution of Eq. (1) has $m = n + k - 1$ zeros in $[a, b)$ then $\eta_k(a) < b$ and an extremal solution of Eq. (1) has the same number of zeros in $[a, \eta_k(a)] \subset [a, b)$. Therefore it is sufficient to prove that for every $\epsilon > 0$, there exists a solution with $n + k - 1$ simple zeros in $[a, \eta_k(a) + \epsilon)$. A priori we select $\epsilon > 0$ such that $\eta_k(a) + \epsilon < \eta_{k+1}(a)$.

Let $y(x)$ be an extremal solution for $[a, \eta_k(a)]$. The system

$$\begin{aligned} u^{(t)}(x_i) &= 0, \qquad 0 \leqslant t \leqslant m(x_i, y) - 1, \qquad i \in I, \quad x_i \neq \eta_k(a),\\ u^{(t)}(\eta_k(a)) &= 0, \qquad 0 \leqslant t \leqslant m(\eta_k(a)) - 3,\\ u^{(t)}(x_j) &= 0, \qquad 0 \leqslant t \leqslant m(x_j) - 2, \end{aligned}$$

of $n - 2$ boundary value conditions has a non-trivial solution $\bar{y}(x)$ which is linearly independent of $y(x)$. As in the proof of Lemma 5, it is easy to see that if $\bar{y}(x)$ has at $\eta_k(a)$ a zero of multiplicity greater than $m(\eta_k(a)) - 2$, then some linear combination of $y(x)$ and $\bar{y}(x)$ will lead to a contradiction. Therefore $\bar{y}(x)$ has at $\eta_k(a)$ a zero exactly of multiplicity $m(\eta_k(a)) - 2$. As in Lemma 5, we find that for suitable α_1, the solution $y_1(x) = y(x) + \alpha_1 \bar{y}(x)$ has at least $n + k - 1$ zeros in $[a, \eta_k(a) + \epsilon)$. Since $\eta_k(a) + \epsilon < \eta_{k+1}(a)$, $y_1(x)$ has exactly $n + k - 1$ zeros in $[a, \eta_k(a) + \epsilon)$. $y_1(x)$ has more simple zeros than $y(x)$, and $M(y_1) < M(y) = n$.

Now, as in the proof of Lemma 4, we define successively $y_2(x) = y_1(x) + \alpha_2 \bar{y}_1(x)$, $y_3(x) = y_2(x) + \alpha_3 \bar{y}_2(x), \ldots$, such that each $y_t(x)$ has exactly $n + k - 1$ zeros in $[a, \eta_k(a) + \epsilon)$ and $y_t(x)$ has more simple zeros than $y_{t-1}(x)$. After a finite number of steps we obtain a solution $y_q(x) = y(x) + \alpha_1 \bar{y}(x) + \cdots + \alpha_q \bar{y}_{q-1}(x)$ which has exactly $n + k - 1$ simple zeros in $[a, \eta_k(a) + \epsilon)$.

When $m(\eta_k(a)) = 1$, we begin the proof by splitting the rightmost multiple zero of $y(x)$. For the intervals of the form (a, b) and $(a, b]$ the proof is similar.

Applying Theorem 2, we shall prove the following:

THEOREM 3. *$\eta_k(\cdot)$ is a strictly increasing continuous function which is defined on an interval of the form $[0, b)$, $0 \leqslant b \leqslant \infty$.*

For the proof we require the following lemmas.

LEMMA 7. *If $\eta_k(a)$ exists, the function η_k is defined and continuous in some neighborhood of a.*

Proof. By Theorem 2 there is a solution $y(x)$ of Eq. (1) with $n + k - 1$ simple zeros in $[a, \eta_k(a) + \epsilon)$, the first of which is at a. Let $u(x)$ be the solution of Eq. (1) which satisfies at c the initial value conditions $u^{(i)}(c) = y^{(i)}(a)$, $i = 0, 1,..., n - 1$. The solutions of Eq. (1) are continuously dependent on the initial conditions. Therefore, if $| c - a |$ is sufficiently small, $y(x)$ and $u(x)$ are close and $u(x)$ has at least $n + k - 1$ zeros in $[c, \eta_k(a) + \epsilon)$, the first of which is c. Thus η_k exists in a neighborhood of a. Moreover, by definition, when $| c - a | < \delta_1$ then $\eta_k(c) < \eta_k(a) + \epsilon$. By interchanging the roles of a and c, we get $\eta_k(a) < \eta_k(c) + \epsilon$ when $| c - a | < \delta_2$. These inequalities prove the continuity of η_k.

LEMMA 8. *If η_k is defined in an interval, it is strictly increasing there.*

Proof. First we show that if η_k is defined at a, then it is strictly increasing in some left neighborhood of a. Indeed, as in Theorem 2, one can show that for given ϵ_1, there is a solution with at least $n + k - 1$ simple zeros in $(a - \epsilon_1, \eta_k(a)]$ and the first of these zeros is in $(a - \epsilon_1, a)$. This solution is given by $v(x) = y(x) + \alpha_0\bar{y}(x) + \alpha_1\bar{y}_1(x) + \cdots + \alpha_m\bar{y}_m(x)$ and as the parameters $\alpha_0, \alpha_1,..., \alpha_m$ vary continuously, its first zero covers some left neighborhood $(a - \epsilon_2, a)$ of a. This means that for every $c \in (a - \epsilon_2, a)$ there is a solution which vanishes at c and has $n + k - 1$ simple zeros in $[c, \eta_k(a)]$. This solution is not an extremal one since it has $n + k - 1$ simple zeros, therefore $\eta_k(c) < \eta_k(a)$.

Now, if a function is continuous in an interval and is strictly increasing in a left neighborhood of each point, it is strictly increasing in the whole interval.

LEMMA 9. *If $\eta_k(a)$ exists, η_k is defined on $[0, a]$.*

Proof. By the proof of Lemma 8, η_k is defined in some open interval A containing a. Let $a' = \inf A$. Then $a' < a$ and η_k is defined in $(a', a]$ and strictly increasing there. In $(a', a]$ we choose a decreasing convergent sequence $a_i \downarrow a'$. Then the decreasing sequence $\eta_k(a_i)$ converges and there is a sequence of extremal solutions $y_i(x)$ such that $y_i(x)$ vanishes at a_i and has $n + k - 1$ zeros in $[a_i, \eta_k(a)]$. We choose a subsequence of $y_i(x)$ which converges together with its derivatives. Its limit function is a solution of Eq. (1) which vanishes at a' and has $n + k - 1$ zeros in $[a', \eta_k(a)]$. Thus $\eta_k(a')$ exists. If $a' > 0$, then by Lemma 7, η_k is defined in a neighborhood of a', contradicting the definition of a'.

This completes the proof of Theorem 3.

In the definition of $\eta_k(a)$ we considered only those solutions of Eq. (1) which vanish at a. Now we see that this restriction in the definition is not necessary. Indeed, assume that there is a solution of Eq. (1) with $n + k - 1$ zeros in $[a, \eta_k(a)]$ such that its first zero in the interval is $c > a$. Then by definition $\eta_k(c) \leqslant \eta_k(a)$ and this contradicts Theorem 3. This observation is stated now as

COROLLARY 4. *No solution of Eq.* (1) *has* $n + k - 1$ *zeros in* $(a, \eta_k(a)]$ *or in* $[a, \eta_k(a))$.

We conclude the paper with the following corollary:

COROLLARY 5. *If* $p(x) > 0$ (*or* $p(x) < 0$), *then* $\eta_k(a)$ *is a continuous function of* $p(x)$.

Proof. Assume $0 \leqslant p(x) - \delta \leqslant p_1(x) \leqslant p(x) + \delta$. Let $y(x)$ be a solution of Eq. (1) which has $n + k - 1$ simple zeros in $[a, \eta_k(a) + \epsilon)$ and $y_1(x)$ the solution of the equation

$$Ly_1 + p_1(x)\, y_1 = 0,$$

which satisfies

$$y_1^{(t)}(a) = y^{(t)}(a), \quad 0 \leqslant t \leqslant n - 1.$$

The solutions of Eq. (1) are continuously dependent on the coefficient $p(x)$. Therefore, for δ sufficiently small, $y_1(x)$ has at least $n + k - 1$ zeros in $[a, \eta_k(a) + \epsilon)$. The continuity of $\eta_k(a)$ as a function of $p(x)$ follows now as in Lemma 7.

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